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Valuing Convertible Bonds with Coupons: A Premium-Decomposition Refinement*

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Abstract

This paper deals with valuing defaultable and non-callable convertible bonds (CBs) with continuous coupon payments. The setup is the Black-Scholes-Merton framework where the underlying firm value evolves according to a geometric Brownian motion. The valuation of CBs can be formulated as an optimal stopping problem, due to the possibility of voluntary conversion prior to maturity. We focus on the notion of premium decomposition, which separates the CB value into the associated European CB value and an early conversion premium. By the Laplace-Carson transform (LCT) approach combined with the premium decomposition, we obtain closed-form LCT solutions for the CB value and the early conversion boundary. They have much simpler expressions than plain LCT solutions without using the premium decomposition. By virtue of the simplicity, we can easily characterize asymptotic properties of the early conversion boundary close or at infinite time to expiry.

1 The PDE Approach

1.1 Assumptions

Following the framework of Merton (1974) and Ingersoll (1977), we consider a CB issued by a firm in frictionless markets, assuming that the CB is the only senior debt in the firm's capital structure except for common stock. Hence, a default would occur when the firm value falls below the total redemption value of the CBs. Let V_t denote the firm value per bond at time t (≥ 0). Assume that $(V_t)_{t\geq 0}$ is a diffusion process with the Black-Scholes-Merton dynamics

$$\mathrm{d}V_t = (r-\delta)V_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t, \quad t \geq 0 \tag{1}$$

where $r > 0$ is the risk-free rate of interest, $\delta > 0$ is the instantaneous rate of the cash payments by the firm to either its shareholders or liabilities-holders (e.g., dividends or interest payments), and $\sigma > 0$ is the volatility coefficient of the firm value, all of which are assumed to be constants. Suppose an economy with finite time period $[0, T]$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a filtration $\mathbb{F} \equiv (\mathcal{F}_t)_{t\in[0,T]}$. $W \equiv (W_t)_{t\in[0,T]}$ is a one-dimensional standard Brownian motion process defined on $(\Omega, \mathcal{F})$ and takes values in $\mathbb{R}$. The filtration $\mathbb{F}$ is the natural filtration generated by W and $\mathcal{F}_T = \mathbb{F}$. The firm value process defined in (1) is represented under the equivalent martingale measure $\mathbb{P}$, which implies that the firm value has mean rate of return r, and the conditional expectation $\mathbb{E}_t[\,\cdot\,] \equiv \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ is calculated under the measure $\mathbb{P}$.

*This is an early draft of my paper Kimura (2018) in preparation. All of the proofs of Lemmas/Theorems and computational results are omitted here.

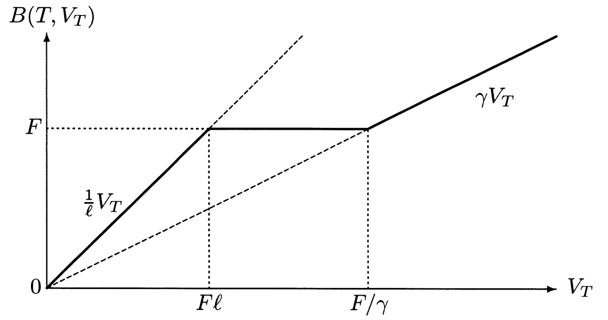

Figure 1: CB payoff at maturity as a function of V_T.

Consider a defaultable CB with maturity date T and face value F. The CB holders receive coupon payments continuously at rate $q > 0$ per unit face value, which means that the holder receives an amount $qF\mathrm{d}t$ in time $\mathrm{d}t$ per bond. For simplicity, we focus on CBs with no call provision and defaultable only at maturity. Assume that there are ℓ outstanding CBs of this firm in markets, and each CB is convertible into n shares. The holders who choose to convert their CBs into shares will dilute current shareholders' ownership. If there are m shares of common stock outstanding, the *conversion value* is given by γV_t, where γ is defined by

$$\gamma = \frac{n}{m + \ell n}, \tag{2}$$

for which $\gamma\ell$ (< 1) is called the *dilution factor*, indicating the fraction of the common stock held by the CB holders.

1.2 PDE for the CB value

Let $B(t, V_t)$ denote the CB value at time $t \in [0, T)$. From the assumptions on the capital structure and the default time, we see that there are three possible payoffs at maturity: CB holders receive either the conversion value γV_T if it exceeds the face value F, the face value F if it exceeds the conversion value γV_T, or the proportional firm value V_T/ℓ if the firm value is less than the par value of outstanding CBs, i.e.,

$$\begin{aligned} B(T, V_T) &= \max\left(\gamma V_T, \ \min\left(\frac{1}{\ell} V_T, F\right)\right) \\ &= \frac{1}{\ell} V_T \mathbf{1}_{\{V_T \le F\ell\}} + F\mathbf{1}_{\{F\ell < V_T \le \frac{F}{\gamma}\}} + \gamma V_T \mathbf{1}_{\{V_T > \frac{F}{\gamma}\}} \\ &= \frac{1}{\ell} V_T - \frac{1}{\ell}\left(V_T - F\ell\right)^+ + \gamma\left(V_T - \frac{F}{\gamma}\right)^+, \end{aligned} \tag{3}$$

where $(x)^+ \equiv \max(x, 0)$ for $x \in \mathbb{R}$. Figure 1 illustrates the payoff value $B(T, V_T)$ at maturity as a function of the firm value V_T. Let $p(V_t)$ be a virtual payoff if the conversion occurs at time t, which is defined by

$$p(V_t) = \frac{1}{\ell} V_t - \frac{1}{\ell}\left(V_t - F\ell\right)^+ + \gamma\left(V_t - \frac{F}{\gamma}\right)^+, \quad 0 \le t \le T. \tag{4}$$

Then, from the theory of arbitrage pricing, the fair CB value at time t is given by solving the *optimal stopping problem*

$$B(t, V_t) = \operatorname*{ess\,sup}_{\tau_c \in [t,T]} \mathbb{E}_t \left[\mathrm{e}^{-r(\tau_c - t)} p(V_{\tau_c}) + \frac{qF}{r} \left(1 - \mathrm{e}^{-r(\tau_c - t)}\right) \right], \quad 0 \le t \le T, \tag{5}$$

where τ_c is a stopping time of the filtration $(\mathcal{F}_t)_{t \in [0,T]}$, and the term $\frac{qF}{r}\left(1 - \mathrm{e}^{-r(\tau_c - t)}\right)$ is the NPV of the coupon payment stream at time t prior to conversion. The random variable $\tau_c^* \in [t, T]$ is called the *optimal conversion time* if it gives the supremum value of the right-hand side of (5).

Let $\mathcal{D} = [0, T] \times \mathbb{R}_+$. Solving the optimal stopping problem (5) is equivalent to finding the points (t, V_t) in $\mathcal{D}$ for which early conversion is optimal. Let $\mathcal{E}$ and $\mathcal{C}$ denote the *early conversion region* and *continuation region*, respectively. The early conversion region $\mathcal{E}$ is defined by

$$\mathcal{E} = \{(t, V_t) \in \mathcal{D} \mid B(t, V_t) = p(V_t)\}.$$

No doubt, the continuation region $\mathcal{C}$ is the complement of $\mathcal{E}$ in $\mathcal{D}$. The boundary that separates $\mathcal{E}$ from $\mathcal{C}$ is referred to as the *early conversion boundary* (ECB), which is defined by

$$V_c(t) = \inf \{V_t \in \mathbb{R}_+ \mid B(t, V_t) = p(V_t)\}, \quad 0 \le t \le T.$$

For simplicity, let $V \equiv V_t$. In much the same way as in the valuation of American options, the value $B(t, V)$ and the ECB $V_c(t)$ can be jointly obtained by solving a *free boundary problem* (Brennan and Schwartz, 1977). From the standard argument of constructing a hedged portfolio consisting of one option and an amount $-\frac{\partial B}{\partial V}$ of the underlying asset, we see that the CB value $B(t, V)$ satisfies an inhomogeneous partial differential equation (PDE)

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 B}{\partial V^2} + (r - \delta) V \frac{\partial B}{\partial V} - rB = -qF, \quad V < V_c(t), \tag{6}$$

together with *boundary conditions*

$$\left| \begin{array}{l} \lim\limits_{V \downarrow 0} B(t, V) = Q_t \\ \lim\limits_{V \uparrow V_c(t)} B(t, V) = \gamma V_c(t) \\ \lim\limits_{V \uparrow V_c(t)} \dfrac{\partial B}{\partial V} = \gamma, \end{array} \right. \tag{7}$$

and a *terminal condition*

$$B(T, V) = p(V), \tag{8}$$

where the boundary value at the origin $V = 0$ is defined by

$$Q_t = \frac{qF}{r}\left(1 - \mathrm{e}^{-r(T-t)}\right), \quad 0 \le t \le T, \tag{9}$$

which is the NPV of the future coupon payment stream at time t prior to maturity. The second condition in (7) is often called the *value-matching condition*, while the third one is called the *smooth-pasting condition*.

1.3 The plain Laplace-Carson transform method

With the change of variables $\tau = T - t$, let

$$\widetilde{B}(\tau, V) = B(T-\tau, V) = B(t, V) \quad \text{and} \quad \widetilde{V}_c(\tau) = V_c(T-\tau) = V_c(t), \quad \tau \geq 0.$$

For $\lambda \in \mathbb{C}$ ($\mathrm{Re}(\lambda) > 0$), define the LCT of these time-reversed functions with respect to τ as

$$B^*(\lambda, V) = \mathcal{LC}[\widetilde{B}(\tau, V)](\lambda) \equiv \int_0^\infty \lambda \mathrm{e}^{-\lambda\tau} \widetilde{B}(\tau, V) \mathrm{d}\tau$$

and

$$V_c^*(\lambda) = \mathcal{LC}[\widetilde{V}_c(\tau)](\lambda) \equiv \int_0^\infty \lambda \mathrm{e}^{-\lambda\tau} \widetilde{V}_c(\tau) \mathrm{d}\tau.$$

Obviously, there is no essential difference between the LCT and the Laplace transform (LT), i.e.,

$$\mathcal{L}[\widetilde{B}(\tau, V)](\lambda) \equiv \int_0^\infty \mathrm{e}^{-\lambda\tau} \widetilde{B}(\tau, V) \mathrm{d}\tau = \frac{B^*(\lambda, V)}{\lambda}, \quad \mathrm{Re}(\lambda) > 0.$$

This relation implies that the LCT can be inverted by using previously established methods for inverting LTs; see Abate and Whitt (1992). Also, for a constant A, the LCT is an identity map, i.e., $\mathcal{LC}[A](\lambda) = A$, whereas $\mathcal{L}[A](\lambda) = A/\lambda$. This invariant property of the LCT is effective to generate much simpler valuation formulas than the LT.

Remark 1. In the context of option pricing, LCTs have been first adopted in the *randomization* of Carr (1998) for valuing an American vanilla put option, of which maturity T is assumed to be exponentially distributed random variable with mean $\mathbb{E}[T] = 1/\lambda$. The idea of randomization gives us another interpretation that the LCT $B^*(\lambda, V)$ can be regarded as an exponentially weighted sum (integral) of the time-reversed value $\widetilde{B}(\tau, V)$ for (infinitely many) different values of the maturity $T \in \mathbb{R}_+$, and hence for $\tau \in \mathbb{R}_+$, which makes LCTs be well defined. From the viewpoint of Carr's randomization, we assume λ is a positive real number.

From the PDE (6) with the conditions (7) and (8), we see that the LCT $B^*(\lambda, V)$ satisfies the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 V^2 \frac{\mathrm{d}^2 B^*}{\mathrm{d}V^2} + (r-\delta) V \frac{\mathrm{d}B^*}{\mathrm{d}V} - (\lambda + r) B^* + \lambda p(V) + qF = 0, \quad V < V_c^*, \tag{10}$$

together with the boundary conditions

$$\left| \begin{array}{l} \lim\limits_{V \downarrow 0} B^*(\lambda, V) = \dfrac{qF}{\lambda + r} \\ \lim\limits_{V \uparrow V_c^*} B^*(\lambda, V) = \gamma V_c^* \\ \lim\limits_{V \uparrow V_c^*} \dfrac{\mathrm{d}B^*}{\mathrm{d}V} = \gamma, \end{array} \right. \tag{11}$$

where we used

$$\mathcal{LC}[\widetilde{Q}_\tau](\lambda) = \frac{qF}{\lambda + r} \quad \text{with} \quad \widetilde{Q}_\tau = Q_{T-\tau} = \frac{qF}{r}\left(1 - \mathrm{e}^{-r\tau}\right).$$

For a given V_c^*, it is straightforward but cumbersome to solve (10) with the boundary conditions (11) and the continuity conditions of $B^*(\lambda, V)$ and its first derivatives at $V = F\ell$, $V = F/\gamma$ and $V = V_c^*$. By this plain LCT approach, we obtain

$$B^*(\lambda, V) = \begin{cases} A_1 V^{\theta_1} + \dfrac{1}{\ell}\dfrac{\lambda V}{\lambda+\delta} + \dfrac{qF}{\lambda+r}, & V \le F\ell \\ A_2 V^{\theta_1} + A_3 V^{\theta_2} + \dfrac{(\lambda+q)F}{\lambda+r}, & F\ell < V \le \dfrac{F}{\gamma} \\ A_4 V^{\theta_1} + A_5 V^{\theta_2} + \gamma\dfrac{\lambda V}{\lambda+\delta} + \dfrac{qF}{\lambda+r}, & \dfrac{F}{\gamma} < V < V_c^* \\ \gamma V, & V \ge V_c^*, \end{cases} \tag{12}$$

where A_i $(i = 1, \ldots, 5)$ are constants given by

$$\begin{aligned} A_1 &= \frac{\lambda\big(\lambda+r+(\delta-r)\theta_2\big)(\gamma^{\theta_1}-\ell^{-\theta_1})F^{1-\theta_1}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)} \\ &\quad - \frac{\theta_2\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_1})F^{1-\theta_2}}{\theta_1(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}(V_c^*)^{\theta_2-\theta_1} + \frac{\delta\gamma}{\theta_1(\lambda+\delta)}(V_c^*)^{1-\theta_1}, \\ A_2 &= \frac{\lambda\big(\lambda+r+(\delta-r)\theta_2\big)\gamma^{\theta_1}F^{1-\theta_1}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)} \\ &\quad - \frac{\theta_2\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_2})F^{1-\theta_2}}{\theta_1(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}(V_c^*)^{\theta_2-\theta_1} + \frac{\delta\gamma}{\theta_1(\lambda+\delta)}(V_c^*)^{1-\theta_1}, \\ A_3 &= -\frac{\lambda\big(\lambda+r+(\delta-r)\theta_1\big)\ell^{-\theta_2}F^{1-\theta_2}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}, \\ A_4 &= -\frac{\theta_2\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_2})F^{1-\theta_2}}{\theta_1(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}(V_c^*)^{\theta_2-\theta_1} + \frac{\delta\gamma}{\theta_1(\lambda+\delta)}(V_c^*)^{1-\theta_1}, \\ A_5 &= \frac{\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_2})F^{1-\theta_2}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}; \end{aligned}$$

see Appendix A for a more unified and simpler expression of these coefficients. The parameters $\theta_1 \equiv \theta_1(\lambda) > 1$ and $\theta_2 \equiv \theta_2(\lambda) < 0$ are two real roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + \big(r - \delta - \tfrac{1}{2}\sigma^2\big)\,\theta - (\lambda + r) = 0. \tag{13}$$

i.e., for $i = 1, 2$,

$$\theta_i(\lambda) = \frac{1}{\sigma^2}\left\{-(r-\delta-\tfrac{1}{2}\sigma^2) - (-1)^i\sqrt{(r-\delta-\tfrac{1}{2}\sigma^2)^2 + 2\sigma^2(\lambda+r)}\right\}. \tag{14}$$

From the value-matching condition in (11), we see that V_c^* satisfies the nonlinear functional equation

$$\frac{\lambda\big(\lambda+r+\theta_1(\delta-r)\big)\big(1-(\gamma\ell)^{-\theta_2}\big)}{\theta_1(\lambda+\delta)(\lambda+r)}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2} + \frac{(1-\theta_1)\delta}{\theta_1(\lambda+\delta)}\frac{\gamma V_c^*}{F} + \frac{q}{\lambda+r} = 0. \tag{15}$$

Note that the existence and uniqueness of the root of (15) will be proved in Section 3; see Theorem 3.

2 Premium-Decomposition Refinement

From the complex solutions (12) and (15), it is really hard to have any prospect of further analysis. To refine these solutions, we will use the notion of premium decomposition: For the CB value $B(t,V)$, we can decompose it into two parts, i.e.,

$$B(t,V) = b(t,V) + \pi(t,V), \quad 0 \le t \le T, \tag{16}$$

where $b(t,V)$ is the value of the European CB associated with the target American CB, and $\pi(t,V)$ is the premium for early conversion. Clearly, $B(t,V)$ and $b(t,V)$ satisfy the common inhomogeneous PDE (6) and they have the same terminal value at $t=T$, i.e.,

$$b(T,V) = B(T,V) = p(V), \tag{17}$$

which is an important key of our refinement; see Remark 3 below for checking $\pi(T,V)=0$.

Furthermore, the European CB value also can be decomposed as

$$b(t,V) = b_0(t,V) + Q_t, \quad 0 \le t \le T, \tag{18}$$

where $b_0(t,V)$ is the corresponding zero-coupon European CB value and Q_t is given in (9). Because of $Q_T = 0$, $b_0(t,V)$ have the same terminal value $p(V)$ as $B(t,V)$ in (17). Hence, applying the risk-neutral valuation method to $b_0(t,V)$, we obtain

$$\begin{aligned} b_0(t,V) &= \mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}p(V)\right] \\ &= \frac{1}{\ell}\mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}V\right] - \frac{1}{\ell}\mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}(V-F\ell)^+\right] + \gamma\mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}\left(V-\frac{F}{\gamma}\right)^+\right] \\ &= \frac{1}{\ell}c(t,V;0) - \frac{1}{\ell}c(t,V;F\ell) + \gamma c(t,V;F/\gamma), \end{aligned} \tag{19}$$

where $c(t,V;K)$ denotes the value of a European vanilla call option with maturity T and strike price K ($K = 0, F\ell, F/\gamma$). The European call value $c(t,V;K)$ has been well known as the Black-Scholes formula, which is given by

$$c(t,V;K) = V\mathrm{e}^{-\delta(T-t)}\Phi\big(d_+(V,K,T-t)\big) - K\mathrm{e}^{-r(T-t)}\Phi\big(d_-(V,K,T-t)\big), \tag{20}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function defined by

$$\Phi(x) = \int_{-\infty}^{x}\phi(y)\mathrm{d}y \quad \text{with} \quad \phi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}, \qquad x \in \mathbb{R},$$

and

$$d_\pm(x,y,\tau) = \frac{\log(x/y) + (r-\delta\pm\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Clearly, $c(t,V;0) = V\mathrm{e}^{-\delta(T-t)}$. With the change of variables $\tau = T-t$, let $\tilde{c}(\tau,V;K) = c(T-\tau,V;K) = c(t,V;K)$ and $\tilde{b}_0(\tau,V) = b_0(T-\tau,V) = b_0(t,V)$. For $\lambda > 0$, define the LCTs $c^*(\lambda,V;K) = \mathcal{LC}[\tilde{c}(\tau,V;K)](\lambda)$ and $b_0^*(\lambda,V) = \mathcal{LC}[\tilde{b}_0(\tau,V)](\lambda)$. Then, from (19), the LCT $b_0^*(\lambda,V)$ can be represented as

$$b_0^*(\lambda,V) = \frac{1}{\ell}\frac{\lambda V}{\lambda+\delta} - \frac{1}{\ell}c^*(\lambda,V;F\ell) + \gamma c^*(\lambda,V;F/\gamma). \tag{21}$$

In order to carry out a further analysis, we need the following lemmas:

Lemma 1.

$$\begin{cases} \lambda + r = -\frac{1}{2}\sigma^2\theta_1\theta_2, \\ \lambda + \delta = -\frac{1}{2}\sigma^2(\theta_1 - 1)(\theta_2 - 1), \\ \lambda + r + \theta_i(\delta - r) = \frac{1}{2}\sigma^2\theta_i(\theta_i - 1), \quad i = 1, 2. \end{cases}$$

Lemma 2.

$$c^*(\lambda, V; K) = \begin{cases} \xi_1(V), & V < K \\ \xi_2(V) + \dfrac{\lambda V}{\lambda + \delta} - \dfrac{\lambda K}{\lambda + r}, & V \geq K, \end{cases}$$

where for $i = 1, 2$,

$$\xi_i(V) \equiv \xi_i(V; K) = \frac{2}{\sigma^2} \frac{\lambda K}{\theta_i(\theta_i - 1)(\theta_1 - \theta_2)} \left(\frac{V}{K}\right)^{\theta_i}.$$

For ease of exposition, for $i = 1, 2$, we write

$$\eta_i \equiv \eta_i(V) = \xi_i(V; F\ell) \quad \text{and} \quad \zeta_i \equiv \zeta_i(V) = \xi_i(V; F/\gamma).$$

Then, from (21) and Lemma 2, we obtain

$$b_0^*(\lambda, V) = \begin{cases} -\dfrac{1}{\ell}\eta_1(V) + \gamma\zeta_1(V) + \dfrac{1}{\ell}\dfrac{\lambda V}{\lambda + \delta}, & V \leq F\ell \\ -\dfrac{1}{\ell}\eta_2(V) + \gamma\zeta_1(V) + \dfrac{\lambda F}{\lambda + r}, & F\ell < V \leq \dfrac{F}{\gamma} \\ -\dfrac{1}{\ell}\eta_2(V) + \gamma\zeta_2(V) + \gamma\dfrac{\lambda V}{\lambda + \delta}, & V > \dfrac{F}{\gamma}. \end{cases} \tag{22}$$

As we saw in Section 2, the LCT $B^*(\lambda, V)$ satisfies the boundary conditions in (11), from which the corresponding boundary conditions for the LCT $\pi^*(\lambda, V) = \mathcal{LC}[\tilde{\pi}(\tau, V)](\lambda)$ for $\tilde{\pi}(\tau, V) = \pi(T - \tau, V) = \pi(t, V)$ can be written as

$$\left| \begin{array}{l} \lim_{V \downarrow 0} \pi^*(\lambda, V) = 0 \\ \lim_{V \uparrow V_c^*} \pi^*(\lambda, V) = \gamma V_c^* - \left(b_0^*(\lambda, V_c^*) + \dfrac{qF}{\lambda + r}\right) \\ \lim_{V \uparrow V_c^*} \dfrac{\mathrm{d}\pi^*}{\mathrm{d}V} = \gamma - \left.\dfrac{\mathrm{d}b_0^*}{\mathrm{d}V}\right|_{V = V_c^*}. \end{array} \right. \tag{23}$$

Since both of $B(t, V)$ and $b(t, V)$ satisfy the common inhomogeneous PDE (6), their difference $B(t, V) - b(t, V) = \pi(t, V)$ satisifies the corresponding *homogeneous* PDE, which means that the LCT $\pi^*(\lambda, V)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 V^2 \frac{\mathrm{d}^2\pi^*}{\mathrm{d}V^2} + (r - \delta)V\frac{\mathrm{d}\pi^*}{\mathrm{d}V} - (\lambda + r)\pi^* = 0, \quad V > 0. \tag{24}$$

From the first boundary condition $\lim_{V \downarrow 0} \pi^*(\lambda, V) = 0$, we have

$$\pi^*(\lambda, V) = A_0 V^{\theta_1}, \quad V \geq 0, \tag{25}$$

where A_0 is a constant. Applying the smooth-pasting condition in (23) to $\pi^*(\lambda, V)$ and using $b_0^*(\lambda, V)$ for $V > F/\gamma$, we obtain

$$A_0 = \frac{1}{\theta_1}\left[\frac{\delta\gamma V_c^*}{\lambda+\delta} + \theta_2\left\{\frac{1}{\ell}\eta_2(V_c^*) - \gamma\zeta_2(V_c^*)\right\}\right](V_c^*)^{-\theta_1},$$

so that for $V < V_c^*$

$$\begin{aligned}\pi^*(\lambda, V) &= \frac{1}{\theta_1}\left[\frac{\delta\gamma V_c^*}{\lambda+\delta} + \theta_2\left\{\frac{1}{\ell}\eta_2(V_c^*) - \gamma\zeta_2(V_c^*)\right\}\right]\left(\frac{V}{V_c^*}\right)^{\theta_1} \\ &= \frac{1}{\theta_1}\left[\frac{\delta\gamma V_c^*}{\lambda+\delta} + \frac{\theta_1-1}{\theta_1-\theta_2}\frac{\lambda F}{\lambda+\delta}\left(1-(\gamma\ell)^{-\theta_2}\right)\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2}\right]\left(\frac{V}{V_c^*}\right)^{\theta_1}. \end{aligned} \tag{26}$$

In addition, from the value-matching condition in (23), we see that the LCT V_c^* satisfies the functional equation

$$\lambda\left(1-(\gamma\ell)^{-\theta_2}\right)\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2} + \theta_2\delta\frac{\gamma V_c^*}{F} + (1-\theta_2)q = 0, \tag{27}$$

which enables us to simplify $\pi^*(\lambda, V)$ in (26) down to

$$\begin{aligned}\pi^*(\lambda, V) &= \frac{1}{\theta_1(\lambda+\delta)}\left[\delta\gamma V_c^* + \frac{\theta_1-1}{\theta_1-\theta_2}\lambda F\left(1-(\gamma\ell)^{-\theta_2}\right)\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2}\right]\left(\frac{V}{V_c^*}\right)^{\theta_1} \\ &= \frac{1}{\theta_1(\lambda+\delta)}\left[\delta\gamma V_c^* + \frac{\theta_1-1}{\theta_1-\theta_2}\{-\theta_2\delta\gamma V_c^* - (1-\theta_2)qF\}\right]\left(\frac{V}{V_c^*}\right)^{\theta_1} \\ &= \frac{(1-\theta_2)\left(\theta_1\delta\gamma V_c^* - (\theta_1-1)qF\right)}{\theta_1(\theta_1-\theta_2)(\lambda+\delta)}\left(\frac{V}{V_c^*}\right)^{\theta_1} \\ &= \frac{2}{\sigma^2(\theta_1-\theta_2)}\left(\frac{\delta\gamma V_c^*}{\theta_1-1} - \frac{qF}{\theta_1}\right)\left(\frac{V}{V_c^*}\right)^{\theta_1}. \end{aligned} \tag{28}$$

Remark 2. It is easy to check the equivalence between two expressions in (15) and (27) for $V_c^*(\lambda)$: Using Lemma 1, we have

$$\begin{aligned}&\frac{\lambda\left(\lambda+r+\theta_1(\delta-r)\right)\left(1-(\gamma\ell)^{-\theta_2}\right)}{\theta_1(\lambda+\delta)(\lambda+r)}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2} + \frac{(1-\theta_1)\delta}{\theta_1(\lambda+\delta)}\frac{\gamma V_c^*}{F} + \frac{q}{\lambda+r} \\ &= \frac{2}{\sigma^2}\left[\frac{\lambda\left(1-(\gamma\ell)^{-\theta_2}\right)}{\theta_1\theta_2(\theta_2-1))}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2} + \frac{\delta}{\theta_1(\theta_2-1)}\frac{\gamma V_c^*}{F} - \frac{q}{\theta_1\theta_2}\right] \\ &= \frac{2}{\theta_1\theta_2(\theta_2-1)\sigma^2}\left[\lambda\left(1-(\gamma\ell)^{-\theta_2}\right)\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2} + \theta_2\delta\frac{\gamma V_c^*}{F} + (1-\theta_2)q\right] = 0,\end{aligned}$$

and hence the coincidence between the equations (27) and (15) can be checked.

Since the zero-coupon European CB value $b_0(t, V)$ is explicitly given in (19) and (20), it would suffice to invert $\pi^*(\lambda, V)$ for obtaining the target CB value $B(t, V)$. Hence, we summarize the results as

Theorem 1. *The value $B(t,V)$ of the CB with voluntary conversion prior to maturity and continuous coupon payments is given by*

$$B(t,V) = \begin{cases} b_0(t,V) + \dfrac{qF}{r}(1 - \mathrm{e}^{-r(T-t)}) + \mathcal{LC}^{-1}[\pi^*(\lambda,V)](T-t), & V < \mathcal{LC}^{-1}[V_c^*(\lambda)](T-t) \\ \gamma V, & V \geq \mathcal{LC}^{-1}[V_c^*(\lambda)](T-t), \end{cases} \tag{29}$$

where $b_0(t,V)$ is the associated zero-coupon European CB value given by

$$b_0(t,V) = \frac{1}{\ell}V\mathrm{e}^{-\delta(T-t)} - \frac{1}{\ell}c(t,V;F\ell) + \gamma c(t,V;F/\gamma),$$

$c(t,V;K)$ is the value of the associated vanilla call option with strike price K ($K = F\ell, F/\gamma$) given by (20), *and*

$$\pi^*(\lambda,V) = \frac{2}{\sigma^2(\theta_1-\theta_2)}\left(\frac{\delta\gamma V_c^*}{\theta_1-1} - \frac{qF}{\theta_1}\right)\left(\frac{V}{V_c^*}\right)^{\theta_1}, \quad V < V_c^*.$$

The LCT $V_c^* \equiv V_c^*(\lambda)$ for the early conversion boundary satisfies the functional equation

$$\lambda(1-(\gamma\ell)^{-\theta_2})\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2} + \theta_2\delta\frac{\gamma V_c^*}{F} + (1-\theta_2)q = 0.$$

Corollary 1.1. *If $\delta = 0$, then it is not optimal for the CB holders to convert early before maturity.*

Corollary 1.2 (Kimura (2017)). *If $q = 0$, then $V_c^*(\lambda)$ is explicitly given by*

$$V_c^*(\lambda) = \frac{F}{\gamma}\left[-\frac{\delta\theta_2}{\lambda(1-(\gamma\ell)^{-\theta_2})}\right]^{\frac{1}{\theta_2-1}}. \tag{30}$$

Theorem 2. *For the time-reversed early conversion boundary $(\widetilde{V}_c(\tau))_{\tau\geq 0}$, we have*

$$\lim_{\tau\to 0}\widetilde{V}_c(\tau) = \lim_{t\to T}V_c(t) = \max\left(1, \frac{q}{\delta}\right)\frac{F}{\gamma}. \tag{31}$$

Remark 3. We see from Theorems 1 and 2 that the desired result $\pi(T,V) = 0$ certainly holds by virtue of the initial-value theorem, i.e.,

$$\pi(T,V) = \lim_{\tau\to 0}\tilde{\pi}(\tau,V) = \lim_{\lambda\to\infty}\pi^*(\lambda,V) = 0,$$

because $V < V_c^*$, $\lim_{\lambda\to\infty}\theta_1(\lambda) = \infty$ and $\lim_{\lambda\to\infty}\delta\gamma V_c^*(\lambda) = \max(q,\delta)F < \infty$.

Theorem 3. *The functional equation* (27) *for the LCT V_c^* has a unique solution larger than* $\max\left(1, \frac{q}{\delta}\right)\frac{F}{\gamma}$.

Theorem 4. *For the time-reversed early conversion boundary* $\left(\widetilde{V}_c(\tau)\right)_{\tau\geq 0}$, *its perpetual value is given by*

$$\lim_{\tau\to\infty}\widetilde{V}_c(\tau)=\lim_{T\to\infty}V_c(t)=\frac{\theta_2^\circ-1}{\theta_2^\circ}\frac{qF}{\delta\gamma}=\frac{\theta_1^\circ}{\theta_1^\circ-1}\frac{qF}{r\gamma},\tag{32}$$

where $\theta_i^\circ\equiv\lim_{\lambda\to 0}\theta_i(\lambda)$ $(i=1,2)$.

Corollary 4.1. *If* $q=0$ *and* $T=+\infty$, *then it is optimal for the CB holders to convert quickly after purchase.*

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A A unified expression for $\{A_i\}$ in Equation (12)

By Lemma 1, the coefficients A_i $(i=1,\ldots,5)$ in (12) can be further simplified in a unified way as follows:

$$\begin{aligned}
A_1&=\frac{2\gamma(V_c^*)^{1-\theta_1}}{\theta_1\sigma^2}\left[\frac{\lambda}{\theta_1-\theta_2}\left\{\frac{1-(\gamma\ell)^{-\theta_1}}{\theta_1-1}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_1-1}-\frac{1-(\gamma\ell)^{-\theta_2}}{\theta_2-1}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2-1}\right\}\right.\\
&\qquad\left.-\frac{\delta}{(\theta_1-1)(\theta_2-1)}\right],\\
A_2&=\frac{2\gamma(V_c^*)^{1-\theta_1}}{\theta_1\sigma^2}\left[\frac{\lambda}{\theta_1-\theta_2}\left\{\frac{1}{\theta_1-1}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_1-1}-\frac{1-(\gamma\ell)^{-\theta_2}}{\theta_2-1}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2-1}\right\}\right.\\
&\qquad\left.-\frac{\delta}{(\theta_1-1)(\theta_2-1)}\right],\\
A_3&=-\frac{2\gamma(V_c^*)^{1-\theta_2}}{\theta_2\sigma^2}\frac{\lambda(\gamma\ell)^{-\theta_2}}{(\theta_1-\theta_2)(\theta_2-1)}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2-1},\\
A_4&=-\frac{2\gamma(V_c^*)^{1-\theta_1}}{\theta_1\sigma^2}\left[\frac{\lambda\left(1-(\gamma\ell)^{-\theta_2}\right)}{(\theta_1-\theta_2)(\theta_2-1)}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2-1}+\frac{\delta}{(\theta_1-1)(\theta_2-1)}\right],\\
A_5&=\frac{2\gamma(V_c^*)^{1-\theta_2}}{\theta_2\sigma^2}\frac{\lambda\left(1-(\gamma\ell)^{-\theta_2}\right)}{(\theta_1-\theta_2)(\theta_2-1)}\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2-1}.
\end{aligned}$$

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